# An Embedding of Existential Graphs into Concept Graphs with Negations

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**Abstract.** Conceptual graphs are based on the existential graphs of Peirce and the semantic networks of AI (see [So92]). Existential graphs are composed of three syntactical elements: lines of identity, predicate names and cuts (which are used for negation). In [Da00] and [Da01] we introduced the cuts of existential graphs as new syntactical element to concept graphs. The resulting *concept graphs with cuts* have at least the expressivity of existential graphs. In this article, we present some ideas how existential graphs can be translated to concept graphs with cuts, or, in other words, how existential graphs can be regarded as special concept graphs with cuts. In order to do this, we provide several examples of existential graphs. We discuss the meanings of these examples and how they should be translated to concept graphs with cuts. After the discussion, we attempt to provide a formal definition of existential graphs and a formal definition of their translation to concept graphs with cuts.

## 1 Introduction

Peirce invented the Existential Graphs (EGs) in 1896. He called them his 'chef d'ouevre' and said they were '*the luckiest find of my career* ' (see [So97]). Peirce also invented the algebraic notation predicate calculus, but he preferred the diagrammatic style of logic. Although the algebraic style of logic became widely accepted, EGs are still relevant. They are used in teaching, in automate reasoning and theorem proving (see for example the works of Hammer and Shin, and John Stewarts PhD-Thesis on theorem proving with EGs). And what is most important for this work: Conceptual graphs are based on EGs. But a solid mathematical foundation of EGs is still missing in literature. In this work we provide an approach of a mathematical foundation of EGs which s based on concept graphs.

In [So97] Sowa says '*Conceptual graphs (CGs) are an extension of existential graphs with features adopted from linguistics and AI.*' The term 'extension' should not be understood syntactically: Sowa adopted the ideas of existential graphs (EGs), but CGs have a different syntax. But 'extension' can be understood semantically: The decisive idea is that everything which can be expressed with EGs can be expressed with CGs, too. A crucial part of this idea is that the cuts of EGs can be expressed by negation contexts in conceptual graphs. But contexts in CGs are handled as *metalevel* operators, but we consider negation as a logical, not a metalevel operator. So we removed the negation contexts from CGs. Instead of them, we have introduced the cuts of EGs as new syntactical element to CGs (see [Da00], [Da01], [Da02]). Furthermore the coreference links are replaced by so-called *identity links*, i.e. edges in the usual sense which are labelled by the identity relation *id*. The resulting graphs are called *concept graphs with cuts*.

We want to stress that Peirce distinguished between diagrams of EGs, which he called *replicas* of EGs, and the graphs themselves. He said: '*A graph is a propositional expression in the System of Existential Graphs of any possible state of the universe. It is a Symbol, and, as such, general, is accordingly to be distinguished from a graph-replica.* ' So a replica of an EG is a diagrammatic representation of an underlying (abstract) EG. Please note that we have the same situation for concept graphs: They can be *represented* diagrammatically, but they *are* no diagrams.

Now the idea that everything which can be expressed with EGs can be expressed with CGs can be refined in the following way: We want to find a mapping $\Xi$[1] which maps an replica of an EG $\mathfrak{E}$ to a concept graphs with cuts $\mathfrak{G} := \Xi(\mathfrak{E})$ which has the same semantical meaning as $\mathfrak{E}$. The mapping $\Xi$ will map lines of identity to concept boxes $\boxed{\top : *}$ and identity links, predicate spots to edges and cuts to cuts.

In this work we want to show how the mapping $\Xi$ should work. In order to do this, we provide several examples of EGs by providing their diagrams. We discuss the meanings of these examples and which difficulties we have to cope with when we want to translate them to concept graphs with cuts. After this discussion, we attempt to provide a formal definition of replicas of EGs as diagrams in the euclidean space. Afterwords we will provide a formal definition of the mapping $\Xi$ which maps the diagrams to special concept graphs with cuts. These concept graphs with cuts can be considered to be the underlying EGs of the replicas. This approach strengthens the mathematical foundations of EGs as well as of EGs, and it shows precisely why and where CGs are an extension of EGs.

Before we start the discussion, we want to give a short overview over the main sources in literature we are referring to. Peirce himself did not write a 'standard textbook' on EGs, and, as Sowa says in his comments in [PS09], '*reading Peirce's manuscripts can be both frustrating and rewarding.*' Roberts worked through Peirce's manuscripts, and his PhD-thesis 'The Existential Graphs of Charles S. Peirce' is a benchmark in the research on EGs and the best introduction in EGs we know. Burch is another expert on the work of Peirce. In his book 'A Peircean Reduction Theses' ([Bu91]) he worked out the 'Peircean Algebraic Logic' which, as he says, '*is designed specifically to accord as closely as possible with the system of Existential Graphs that Peirce developed in the late 1890s.*'

[1] We have chosen the letter $\Xi$ for two reasons: First, we have decided to use a capital greek letter following the well known mapping $\Phi : CG \to FOL$ (and the mapping $\Psi : FOL \to CG$ which some authors use, too). Secondly, the form of '$\Xi$' is similar to 'E', the first letter of 'Existential Graphs'.

In order to understand the graphical representations of EGs, chapter 11 of his book is very instructive. Sowa provides in his manuscript 'Logic: Graphical and Algebraic' ([So97]) a short introduction into EGs. Furthermore he has written enlightening comments on MS514, which Peirce wrote in 1909 as a tutorial on EGs. MS514 is also one of two work sources of Peirce we use for our analysis of EGs. The other one is given by his Cambridge lectures from 1898 (esp. Lecture 3: 'The Logic of Relatives'). These are the main sources we use in this article. Of course the mentioned authors have written more on EGs, and there are more authors which are experts on EGs (e.g. Hammer and Shin).

## 2 Examples for Existential Graphs

In this section we provide some examples for EGs and their translation to concept graphs with cuts.
EGs are composed of three kinds of parts:

- lines of identity, which are used to denote the existence of objects and the identity between objects (in this work, we will write 'LoI' instead of 'line of identity' for short)
- predicate names, which are attached to LoIs and which are used to denote attributes of or relations between the objects
- cuts, which are used to denote negation. In his later work, e.g. in [PS09], Peirce used shaded areas instead of cuts. In this work, we use cuts to keep conform with the notation we used in [Da00], [Da01] and [Da02]. But when we adopt an example Peirce has given in [PS09], we draw them in their original manner with shaded and unshaded areas instead of cuts.

**1st Example:**
The first EG $\mathfrak{E}_1$ we want to discuss is a single line of identity, i.e. $\mathfrak{E}_1 :=$ ——
The (naive) meaning of this graph is 'something exists', or, perhaps better, 'there is something'.
LoIs are so-called 'indivisible graphs'. Although LoIs are indivisible, they bear a kind of inner structure. In [PS09] Peirce writes: '*The line of identity can be regarded as a graph composed of any number of dyads '-is-' or as a single dyad.*' He illustrates this view with an example (page 14 in [PS09]). According to this, we can regard $\mathfrak{E}_1$ in different ways. If we regard $\mathfrak{E}_1$ as composed of three dyads, we get the meaning 'there is something which is something which is something which is something'. If we regard this LoI as a single dyad, we get the meaning 'there is something which is something'. But note that Peirce does not mention that a LoI can be regarded as a monad (which would yield the meaning 'there is something').
In his commentary of this part of [PS09], Sowa provides the following example: *man—is—is—is—is—will die* . He explains its translation to an FOL-formula in the following way: '*Each of the five segments of the line of identity corresponds*

*to an existentially quantified variable, and each instance of the dyad –is– corresponds to an equal sign between two variables.*' Hence Sowa adopted Peirce's view on LoIs.

Burch shares this understanding as well. In [Bu91] he describes his comprehension of LoIs: '*Lines of identity are simply lines that are themselves composed of spots of identity (of various adicities) that are directly joined together.*' The spots of identity correspond to the existentially quantified objects (the 'somethings' in Peirce's translations resp. the existentially quantified variables), and their joins correspond to the relation 'is' resp. the equal sign between two variables.

Roberts makes in [Ro73] a similar approach. He provides three rules C7, C8 and C9 for the reading of LoIs in EGs. The first of these rules is: '*C7: A heavy line, called a line of identity, shall be a graph asserting the numerical identity of the individuals denoted by its two extremities.*' This rule expresses Peirce's understanding that a LoI can be regarded as a single dyad.

To summarize: The translation of $\mathfrak{E}_1$ to an FOL-formula or to a concept graph depends on the number of dyads '-is-' (or of one monad) it is composed of. This number is our choice. So we have infinite many different possible translations of $\mathfrak{E}_1$, namely the following:

| | one monad | one dyad | two dyads | ... |
|---|---|---|---|---|
| FOL | ? | $\exists x_1.\exists x_2.x_1 = x_2$ | $\exists x_1.\exists x_2.\exists x_3.x_1 = x_2 \wedge x_1 = x_2$ | ... |
| CG | $\boxed{\top:*}$ | $\boxed{\top:*}$ — ( = ) — $\boxed{\top:*}$ | $\boxed{\top:*}$ — ( = ) — $\boxed{\top:*}$ — ( = ) — $\boxed{\top:*}$ | ... |

Obviously all FOL-formulas are (semantically) equivalent (esp. all formulas are equivalent to $\exists x.x = x$). We will now explain why the same holds for the concept graphs as well.

In [Da01], we provided a sound and complete calculus for concept graphs with cuts. This calculus contains esp. the rule **splitting a vertex**. This rule may be reversed (i.e. the rule is a transformation which may be performed in both directions). The reverse direction is named **merging two vertices** (see [Da02]). These rules do the following: [2]

- **merging two vertices**
  Let $e \in E^{id}$ be an identity link with $\nu(e) = (v_1, v_2)$ and $cut(v_1) \geq cut(e) = cut(v_2)$. Then $v_1$ may be merged into $v_2$, i.e. $v_1$ and $e$ are erased and, for every edge $e \in E$, $e(i) = v_1$ is replaced by $e(i) = v_2$.
- **splitting a vertex**
  Let $v = \boxed{\top:*}$ be a vertex in cut $c_0$ and incident with relation edges $R_1, \ldots, R_n$, placed in cuts $c_1, \ldots, c_n$, respectively. Let $c$ be a cut such that $c_1, \ldots, c_n \leq c \leq c_0$. Then the following may be done: In $c$, a new vertex $v' = \boxed{\top:*}$ and a new identity-link between $v$ and $v'$ is inserted. On $R_1, \ldots, R_n$, arbitrary instances of $v$ are substituted by $v'$.

These two rules allow to insert and erase redundant copies of concept boxes $\boxed{\top:*}$. These rules are for concept graphs with cuts what Peirce's view that a LoI can

[2] For a further discussion and examples see [Da01] and [Da02]

be regarded as a graph composed of any number of dyads '-is-' is for EGs. In other words: These rules are exactly the rules which are needed to see that all the different possible translations of an EG are semantically equivalent. So we set:

**Definition 1.** *Let $\theta$ be the smallest equivalence relation such that the following holds: If $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are two concept graphs with cuts and we can derive $\mathfrak{G}_2$ from $\mathfrak{G}_1$ only with the rules* splitting a vertex *and* merging two vertices[3]*, then we have $\mathfrak{G}_1 \theta \mathfrak{G}_2$. We say that $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are $\theta$-equivalent.*

From the discussion so far we can draw the following conclusion: All the possible correct translations of $\mathfrak{E}_1$ are $\theta$-equivalent. So it appears reasonable that the translation of an EG is not a single concept graph, but a whole equivalence class of $\theta$. The mapping $\Xi$ has to yield a single member of this class.

An obvious approach for $\Xi$ is to assign *one* concept box $\boxed{\top{:}*}$ to each LoI. But in Example 10 it will turn out that this approach fails. So we will appeal to understand that a LoI is a single dyad which asserts the identity between the two endpoints of the LoI. So we assign a concept box $\boxed{\top{:}*}$ to each endpoint of the LoI and an identity link between them, i.e. we set $\Xi(\mathfrak{E}_1) :=$ $\boxed{\top{:}*}$—(=)—$\boxed{\top{:}*}$.

**2nd and 3rd Example:**

In the next two examples we introduce predicates to our discussion. We start with an example having a unary predicate $P$ which is attached to a LoI. The EG is therefore

$$\mathfrak{E}_2 := P\text{———}$$

In this example we have one LoI. On one of its end points a predicate name is attached. In the following, we will denote such points *predicate spots*.

When we translate EGs to concept graphs, it is obvious that predicate spots should be translated to edges, i.e. relations, in concept graphs. We have argued that we will translate a LoI to *two* concept boxes $\boxed{\top{:}*}$ which are connected with an identity link. One might think that in the translation of $\mathfrak{E}_2$ we can drop the concept box $\boxed{\top{:}*}$ which is the translation of the predicate spot of our EG (i.e. the endpoint of the line of identity where the predicate name $P$ is attached). This means we would assign concept boxes $\boxed{\top{:}*}$ only to 'loose' ends of LoIs. Doing the translation this way, our translation of $\mathfrak{E}_2$ would be (P)—$\boxed{\top{:}*}$ .

But the next graph shows that this approach fails: $\mathfrak{E}_3 := P\text{———}Q$

The translation of $\mathfrak{E}_3$ must contain at least one concept box $\boxed{\top{:}*}$ and two unary edges with the predicate names $P$ and $R$. But the LoI of $\mathfrak{E}_3$ has no loose end. Hence it is generally not sufficient to assign concept boxes $\boxed{\top{:}*}$ only to loose ends of LoIs.

Sowa says in [PS09] the following: '*In EGs, each predicate is represented by a character string [...] and each argument or subject is represented by a line called a peg. By itself, a peg* [4] *represents an existentially quantified variable, and a LoI that connects two or more pegs corresponds to an equal sign '=' between the*

[3] for technical reasons, the rule *isomorphism* is needed, too.

[4] The term 'hook' instead of 'peg' is used by Peirce, too.

*corresponding variables.*' The crucial conclusion we can draw from this statement is that we have to assign a concept box $\boxed{\top{:}*}$ (which is the correspondent of 'existentially quantified variable') in concept graphs to *each peg* of a predicate spot. Hence we set

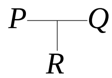
$$\Xi(\mathfrak{C}_2) := (P)\!-\!\boxed{\top{:}*}\!-\!(=)\!-\!\boxed{\top{:}*} \qquad \text{and} \qquad \Xi(\mathfrak{C}_3) := (P)\!-\!\boxed{\top{:}*}\!-\!(=)\!-\!\boxed{\top{:}*}\!-\!(Q)$$

These graphs contain, similar to our first example, a number of redundant copies of $\boxed{\top{:}*}$. But we have

$$\Xi(\mathfrak{C}_2)\,\theta\ (P)\!-\!\boxed{\top{:}*} \qquad \text{and} \qquad \Xi(\mathfrak{C}_3)\,\theta\ (P)\!-\!\boxed{\top{:}*}\!-\!(Q)$$

**4th Example:**

Now we consider a graph in which we have a linked structure of lines.

$\mathfrak{C}_4 :=$ 

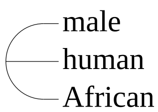

There are different possibilities how this line with several branches can be seen. Roberts explains a similar example in his book: '*We could consider the [...] lines as a single line of identity with three extremities which have a point in common [...]. And the totality of all the lines of identity that join one another he* (Peirce) *called a 'ligature'. we prefer the former terminology [...]*'

Later on he explains how a branching LoI should be treated. This is the next of his three rules we mentioned in our first example. He states: '*C8: A branching line of identity with n number of branches will be used to express the identity of the n individuals denoted by its n extremities.*'

Sowa shares the understanding that the linked structure can be regarded as a single LoI. For example, in [So97] he says: '*In Peirce's graphs, a bar or a linked structure of bars is called a line of identity.*'

Peirce's understanding changes among different manuscripts. In his Cambridge lectures of 1898 we find the phrase: '*Now as long as there is but one such line of identity, whether it branches or not [...].*'

But in his tutorial [PS09] of 1909 he he provides an example which he explains as follows:




*is a graph instance composed of instances of three indivisible graphs which assert 'there is a male', 'there is something human' and 'there is an African'. The syntactic junction or point of teridentity asserts the identity of something denoted by all three.*

Later on he says: '*A line which is composed of two or more lines of identity abutting on one another is called a 'ligature'.*' So he explicit discriminates between *one* line of identity and a linked structure of lines of identity which he calls ligature.

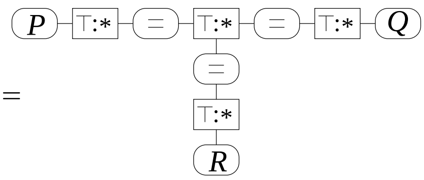
Our understanding of $\mathfrak{C}_4$ is the following: $\mathfrak{C}_4$ has three LoIs. Each of them has one endpoint which is a predicate spot. We consider the 'syntactic junction', the 'point of teridentity' as a point which have all the LoIs in common, i.e. the three

LoIs share a common endpoint. Following Burch, we will call points like this *identity spots*. As long as a ligature does not cross a cut (a further discussion on this will follow in Example 8), it makes no semantical difference whether we understand the ligature in $\mathfrak{E}_4$ as composed of three LoIs or as being a single LoI with three branches. We prefer the former view due to mere technical reasons: With this view it is easier to provide a mathematical definition for EGs and to provide a formal translation of EGs to concept graphs (see next section).

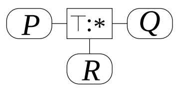
Before we proceed with this example, we refine our informal definition of predicate spots, identity spots and pegs resp. hooks as follows:

- A *predicate spot* is a point where a predicate name is scribed. We presuppose that in each predicate spot ends at least one LoI.
- If a predicate spot carries a predicate name with arity $n$, there will be exactly $n$ endpoints of LoIs attached to this predicate point. This $n$ endpoints are called *pegs* or *hooks*.
- An *identity spot* is an endpoint of a LoI which is not a predicate spot.

Using this terminology, $\mathfrak{E}_4$ has three LoIs (which yield three identity links in our translation) and three predicate spots (which yield three further edges in our translation). Each predicate spot carries one peg, and we have a further identity spot, hence we will have four concept boxes $\boxed{\top:*}$ in our translation. This yields $\Xi(\mathfrak{E}_6)$.

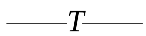
$\Xi(\mathfrak{E}_4) :=$ 


$\Xi(\mathfrak{E}_4)$ contains again a number of redundant concept boxes $\boxed{\top:*}$. But like in the last examples, we have a uniquely given concept graph which is equivalent to our translation and which has a minimal number of concept boxes $\boxed{\top:*}$. We have

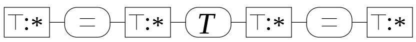
$\Xi(\mathfrak{E}_4)\,\theta$ 

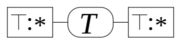

**5th Example:**

If we use predicates with an arity $> 1$, the EGs can be read the same way. We start with the following example, having a dyadic predicate $T$:

$\mathfrak{E}_5 :=$ 

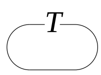

$\mathfrak{E}_5$ has two LoIs, one predicate spot with two pegs and two identity spots. Hence

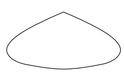
$\Xi(\mathfrak{E}_5) :=$ 

$\theta$ 

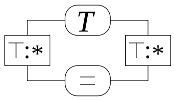

**6th and 7th Example:**

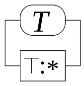
The following graphs have only one LoI. In $\mathfrak{E}_6$, its extremities are attached to the two pegs of the dyadic predicate $T$. In $\mathfrak{E}_7$ they are simply joined.

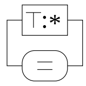
$\mathfrak{E}_6 :=$ 

and $\mathfrak{E}_7 :=$ 

The main difference between these two examples is the following: In $\mathfrak{E}_6$, we have two pegs to which we will assign two concept boxes $\boxed{\top:*}$ in our translation to concept graphs. In $\mathfrak{E}_7$, we have only one identity spot, hence we will have only one concept box $\boxed{\top:*}$ in our translation. So we have

$\varXi(\mathfrak{E}_6) :=$ 

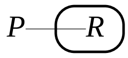
$\theta$ 

and $\varXi(\mathfrak{E}_7) :=$ 


But it is worth to note that $\varXi(\mathfrak{E}_7)$ is *not* $\theta$-equivalent to $\boxed{\top{:}*}$ !

**8th Example:**

Finally we have to introduce cuts to our discussion. We start with a graph in which a LoI seems to cross a cut.

$\mathfrak{E}_8 :=$ 


We find the phrase ‘a line of identity crossing a cut’ several times in the book of Roberts. Sowa shares the understanding that a LoI may cross a cut with Roberts.

In his commentary in [PS09] he explains the graph on the right as follows: ‘*[...] part of the line of identity is outside the negation. When a line of identity crosses one ore more negations [...]*’


But in [PS09] Peirce offers a different point of view. In our first example, we have cited Peirce’s definition of LoIs. Here is the whole quotation: ‘*Every indivisible graph instance must be wholly contained in a single area. The line of identity can be regarded as a graph composed of any number of dyads ‘-is-’ or as a single dyad. But it must be wholly in one area. Yet it may abut upon another line of identity in another area.*’ Especially we can conclude that Peirce did not allow LoIs to cross a cut[5]. To emphasize this, Peirce provides the following example, which he describes as follows:

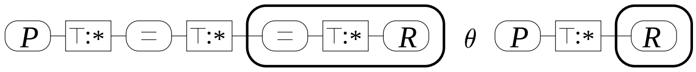

*Thus it denies that there is a man that will not die, that is, it asserts that every man (if there be such an animal) will die. It contains two LoIs (the part in the shaded area and the part in the unshaded area).*

So our interpretation of $\mathfrak{E}_8$ is the following: $\mathfrak{E}_8$ contains *two* LoIs. In the words of Peirce, ‘they abut on one another’. Our understanding is that they have one point, an identity spot, in common. This identity spot is placed on the cut.

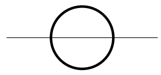
We have to analyze how points on a cut have to be treated. In his PhD-thesis, Roberts cites Peirce as follows: ‘*The cut is outside its own close.*’ From this, he derives the last rule ‘*C9: Points on a cut shall be considered to lie outside the area of that cut.*’ We adopt this view and draw from this the following conclusions for our translation of $\mathfrak{E}_8$:

In this translation, we assign a concept box $\boxed{\top{:}*}$ to the identity spot on the cut, and this box is placed *outside* the cut. All the remaining spots of the LoI are placed inside the cut. So the concept box $\boxed{\top{:}*}$ we assign to the peg of $R$ is placed *inside* the cut. The same holds for the identity link between these two boxes which we assign to the right LoI: It is placed inside the cut, too. The left LoI of $\mathfrak{E}_8$ is easier to understand. The concept box $\boxed{\top{:}*}$ we assign to its left endpoint (i.e. the peg of $P$) and the identity link we assign to the LoI have obviously to be placed *outside* the cut. So we get

[5] Burch pointed this out in his talk on ICCS 2001, too.

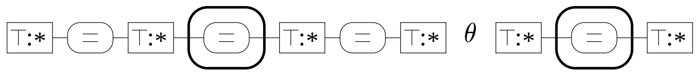
$\Xi(\mathfrak{E}_8) :=$ 


**9th Example:**

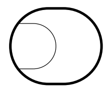
A well known example is the following graph (see Figure 13 on page 53 in [Ro73]): $\mathfrak{E}_9 :=$ 

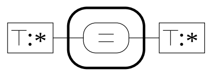
The meaning of this graph is 'there are at least two things' or, as Roberts says in [Ro73]: '*This devise signifies the non-identity of the individuals denoted by the extremities of the ligature: 'There are two objects such that no third object is identical to both'.*' In particular Roberts interprets the graph in the following way: It contains a ligature which is composed of three LoIs, and each LoI corresponds to one object. If we adopt this interpretation for the mapping $\Xi$, we would assign a concept box $\boxed{\top:*}$ to each LoI (instead of assigning concept boxes $\boxed{\top:*}$ to *endpoints* of LoIs). But the next example shows that this approach may fail. So our understanding of $\mathfrak{E}_9$ is the following:

$\mathfrak{E}_9$ contains a ligature which is composed of three LoIs. The LoI in the middle has with each of the other two LoIs an identity spot in common, and these two spots are placed *on* the cut. Hence the concept boxes we assign to these spots are placed *outside* the cut. As the remaining identity spots of the LoI in the middle are placed *inside* the cut, the identity link which we assign to this LoI is placed *inside* the cut, too. The concept boxes we assign to the extremities of the ligature and the identity links we assign to the left and right LoI have to be placed *outside* the cut. This yields altogether

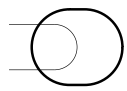
$\Xi(\mathfrak{E}_9) :=$ 


**10th Example:**

This example is closely related to the last one. But $\mathfrak{E}_{10}$ contains only one LoI (which corresponds the the LoI in the middle of $\mathfrak{E}_9$). Both endpoints are placed on the cut. $\mathfrak{E}_{10} :=$ 

Like in the last example, the concept boxes $\boxed{\top:*}$ we assign to the endpoints are placed outside the cut, and the identity link between these boxes is placed *inside* the cut. Hence the translation of this EG graph is: $\Xi(\mathfrak{E}_{10}) :=$ 

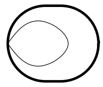

So $\mathfrak{E}_9$ and $\mathfrak{E}_{10}$ are semantically equivalent. This is not surprising: Roberts provides in [Ro73] two examples (Figure 3 and 4 on page 54) on which he explains how LoIs which '*terminate on a cut* ' (Roberts) have to be treated. According to this, $\mathfrak{E}_{10}$ is equivalent to the graph on the right, which is another way of drawing $\mathfrak{E}_9$.



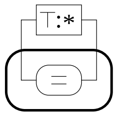
But we want to stress that each concept graph which has the same meaning as $\mathfrak{E}_9$ or $\mathfrak{E}_{10}$ needs at least two concept boxes $\boxed{\top:*}$! So $\mathfrak{E}_{10}$ cannot be translated to a concept graph with only *one* concept box, although $\mathfrak{E}_{10}$ has only one LoI. $\mathfrak{E}_{10}$ is the crucial example why we assign one concept box $\boxed{\top:*}$ to *each extremity* of a LoI, and why we do not assign one concept box $\boxed{\top:*}$ to *each LoI itself.*

**11th Example:**

Now we consider the EG on the right. Again we have only one LoI, but the endpoints of this LoI are identical.

$\mathfrak{E}_{11} :=$ 

Contrary to the last example, we have *one* identity spot instead of two. So our translation of $\mathfrak{E}_{11}$ contains only *one* concept box $\boxed{\top{:}*}$. This yields the translation on the right.

$\Xi(\mathfrak{E}_{11}) :=$ 


**12th Example:**

In the following we want to analyze some examples in which LoIs seem to 'touch' a cut. In [PS09] Peirce demonstrates the inference of a syllogism with EGs. In this demonstration he provides the two EGs on the right (numbering taken from Peirce):

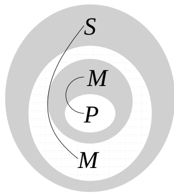


Fig. 12

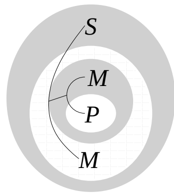
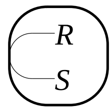


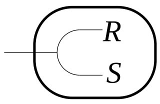
Fig. 13

He derives the EG of Fig. 13 with the insertion rule from the EG of Fig. 12. These graphs show that Peirce had indeed the concept of 'lines of identity touching a cut' and how he treats them.

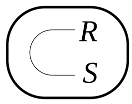
The first example we want to consider is $\mathfrak{E}_{12}$. The spot where the LoI in $\mathfrak{E}_{12}$ touches the cut is in our view a point which is placed *on* the cut and has therefore be considered outside the cut.

$\mathfrak{E}_{12} :=$ 

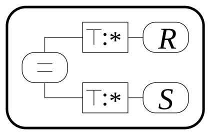

According to this, the graph on the right has the same meaning as $\mathfrak{E}_{12}$:




But this graph has a *different* meaning than $\mathfrak{E}_{12}$:



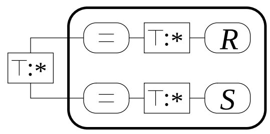

If we assume that $\mathfrak{E}_{12}$ has only one LoI which touches the cut in its middle, we would (according to our translation rules given so far) translate this graph to the graph on the right. This concept graph is *not* an appropriate translation of $\mathfrak{E}_{12}$.

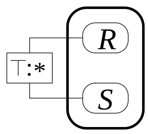



So if we insist on the interpretation that this graph has *one* LoI, we would have take into account that there are cases where we have to assign concept boxes $\boxed{\top{:}*}$ to identity spots which are located in the middle of a LoI. This would make a formal translation of EGs to concept graphs with cuts more complicated.

For this reason, it is better to understand the rule 'lines of identity do not cross cuts' strictly in the following sense: We only allow *end*points of LoIs to be placed directly on a cut.

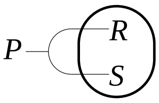
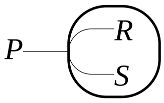
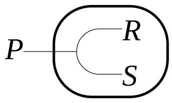
According to this view, $\mathfrak{E}_{12}$ has *two* LoIs which have an identity spot in common, and this identity spot is placed on the cut. This yields

$\Xi(\mathfrak{E}_{12}) :=$ 

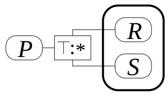
$\theta$ 

**13th, 14th and 15th Example:**

How the touching LoI of $\mathfrak{E}_{12}$ can be seen is elaborated in the next three examples. Each graph has a ligature with three branches and a teridentity spot which, according to rule C8 of Roberts, expresses the identity of the three attached LoIs.

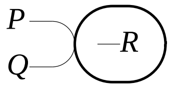
$\mathfrak{E}_{13} :=$  , $\mathfrak{E}_{14} :=$  , $\mathfrak{E}_{15} :=$ 

It turns out that $\Xi$ maps $\mathfrak{E}_{13} - \mathfrak{E}_{15}$ to the same $\theta$-class, namely the class of the graph on the right. In particular $\mathfrak{E}_{13} - \mathfrak{E}_{15}$ have the same meaning.

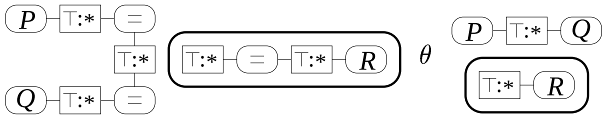


**16th Example:**

Now we give an example of a graph in which a LoI seems to touch from the outside. We consider the following EG:

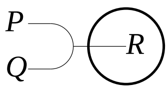
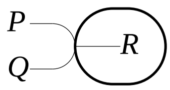
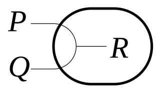
$\mathfrak{E}_{16} :=$ 

$\mathfrak{E}_{16}$ has three LoIs. Two of them have an identity spot on the cut in common. So $\mathfrak{E}_{16}$ has three predicate spots, each of them has one peg, and two identity spots. So we have

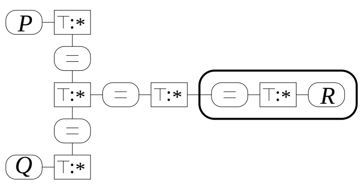
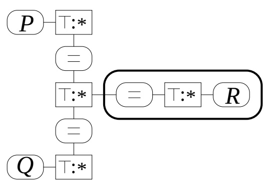
$\Xi(\mathfrak{E}_{16}) :=$ 

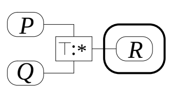
**17th, 18th and 19th Example:**

Like for $\mathfrak{E}_{12}$ we want to elaborate how the touching LoI of $\mathfrak{E}_{16}$ can be treated.

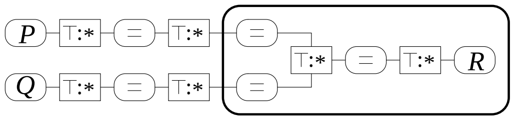
$\mathfrak{E}_{17} :=$  , $\mathfrak{E}_{18} :=$  , $\mathfrak{E}_{19} :=$ 

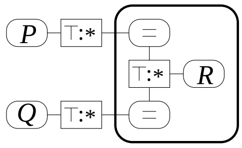
Compare $\mathfrak{E}_{18}$ with the EG of Fig. 13 in [PS09]. In $\mathfrak{E}_{18}$ the teridentity spot is placed on the cut and can therefore, according to rule C9, be considered to lie outside the cut. So it has to be expected that $\mathfrak{E}_{17}$ and $\mathfrak{E}_{18}$ have the same meaning. We have indeed

$\Xi(\mathfrak{E}_{17}) :=$  $\theta$  $=: \Xi(\mathfrak{E}_{18})$

Note that both translations are equivalent to 

But we have

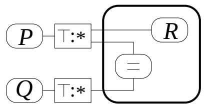
$\Xi(\mathfrak{E}_{19}) :=$ 

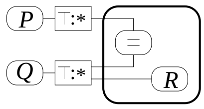
$\theta$ 


Here are two aspects remarkable.

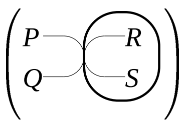
We want to point out that $\mathfrak{E}_{19}$ has a different meaning than $\mathfrak{E}_{17}$ and $\mathfrak{E}_{18}$. More precisely: $\mathfrak{E}_{17}$ and $\mathfrak{E}_{18}$ entail $\mathfrak{E}_{19}$, but not vice versa.

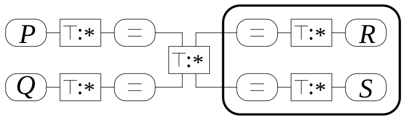
$\mathfrak{E}_{19}$ is furthermore our first example where the class of all concept graphs which are $\theta$-equivalent to $\Xi(\mathfrak{E}_{19})$ does *not* contain a uniquely given element with a minimal number of concept boxes $\boxed{\top:*}$. For $\mathfrak{E}_{19}$ we have *two* minimal graphs which are not isomorphic, namely

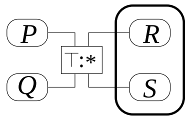


, 


**20th Example:** Finally we want to remark that

$\Xi\Bigg($ 

$\Bigg) :=$ 

$\theta$ 


## 3 Definitions

In this section we attempt to provide a formal definition of EGs and a formal definition of $\Xi$. For the formal definition of concept graphs with cuts see [Da01] or [Da02].

We want to note that Peirce's understanding of EGs depends on his understanding of the continuum, and this understanding is very different from the set $\mathbb{R}$. For this reason we needed to discuss the semantics of several 'bordercases' of EGs (for example: touching LoIs). Nevertheless we provide a mathematization of EGs as a structure of lines and curves in $\mathbb{R}^2$ because $\mathbb{R}^2$ is the standard mathematization of the euclidean plane. So 'formal replicas of EG' can be understood to be defined as closely as possible to Peirce's replicas of EGs in contemporary mathematics.

**Definition 2.** *Let* $\mathcal{R} := (\mathcal{R}_i)_{i\in\mathbb{N}}$ *be a family of finite sets* $\mathcal{R}_i$ *whose elements are called* relation names. *The elements of* $\mathcal{R}_i$ *have the arity* $i$.

*A* formal replica of an existential graph over $\mathcal{R}$ *is a structure* $(L, (\nu_l)_{l\in L}, \top, Cut, (\nu_c)_{c\in Cut}, P, (\nu_p)_{p\in P})$ *where*

– $L$, $Cut$, $P$ *are disjoint finite sets which are called* lines of identity, cuts *and* predicate spots*, resp.,*
– $\top$ *is a single element, the* sheet of assertion*,*
– *each* $\nu_l$*,* $l \in L$ *is a differentiable function* $\nu_l : [0,1] \to \mathbb{R}^2$ *such that for* $x, y \in [0,1]$ *with* $\nu_l(x) = \nu_l(y)$ *we have* $x = y$ *or* $\{x, y\} = \{0, 1\}$

- *each $\nu_c$, $c \in Cut$ is a differentiable and injective function $\nu_c : S^1 \to \mathbb{R}^2$, where $S^1$ is the circle in the euclidean plane with radius 1 and center $\vec{0}$.*
- *each $\nu_p$ is a structure $(R_p, \vec{x}_p, (l_{p,i}, x_{p,i})_{i=1,\ldots,k})$ with $R_p \in \mathcal{R}_k$, $\vec{x}_p \in \mathbb{R}^2$ and $l_{p,i} \in L, x_{p,i} \in \{0,1\}$ for $i = 1,\ldots,k$ (and we set $k := arity(p) := arity(R_p)$)*

*such that the following conditions hold:*

- **Intersection conditions**
  *Let $\nu_m(x) = \nu_n(y)$ for $m, n \in L \cup Cut$. Then we have*
  - *$\{m,n\} \cap L \neq \emptyset$ (i.e. cuts do not intersect)*
  - *$m \in L, n \in Cut \Longrightarrow x \in \{0,1\}$ and $m \in Cut, n \in L \Longrightarrow y \in \{0,1\}$*

  *We further suppose that $\{((m,x),(n,y)) \mid \nu_m(x) = \nu_n(y), m,n \in L \cup Cut\}$ is finite.*
- **Predicate Spots conditions**
  - *For each predicate spot $\nu_p := (R_p, \vec{x}_p, (l_{p,i}, x_{p,i})_{i=1,\ldots,arity(p)})$ we have*
    - *$\nu_{l_{p,i}}(x_{p,i}) = \vec{x}_p$ for $i = 1,\ldots,arity(p)$*
    - *If $\nu_l(x) = \vec{x}_p$ with $l \in L$ and $x \in [0,1]$, then $(l,x) \in \{(l_{p,i}, x_{p,i}) \mid i = 1,\ldots,arity(p)\}$*
    - *$i \neq j$ implies $(l_{p,i}, x_{p,i}) \neq (l_{p,j}, x_{p,j})$*
  - *If we have two predicate spots $p \neq q$, then we have $\vec{x_p} \neq \vec{x_q}$*
  - *For each predicate spot $p$ there is no cut $c \in Cut$ with $\vec{x_p} \in \nu_c[S^1]$*

Before we define $\Xi$, we first need some auxiliary definitions. For this let $\mathfrak{E} := (L, (\nu_l)_{l \in L}, \top, Cut, (\nu_c)_{c \in Cut}, P, (\nu_p)_{p \in P})$ be an EG.
Let $c \in C$ be a cut. The Jordan Curve Theorem yields that $\nu_c$ partitions the plane into two disjoint connected components, one of which is bounded and one not bounded. We denote the bounded component with $in(c)$ and the unbounded component with $out(c)$, and we assume that the cut itself belongs to $out(c)$ (i.e. $\nu_c[S^1] \subseteq out(c)$). For the sheet of assertion we set $in(\top) := \mathbb{R}^2$ and $out(\top) := \emptyset$.
Cuts may contain each other (see for example Fig. 12 and 13 in [PS09]). This induces canonically an order $\leq$ on $Cut \cup \{\top\}$, which now can be defined as follows: for $c, d \in Cut \cup \{\top\}$ we set $c < d :\Longleftrightarrow \nu_c[S^1] \subseteq in(d)$.
Note that for $c \in Cut \cup \{\top\}$, $in(c)$ is the set of all points of the plane which are enclosed by $c$, even if they are deeper nested inside other cuts. The points of $in(c)$ which are *not* deeper nested inside other cuts are said to by *directly* enclosed. The set of all directly enclosed points is the *area* of a cut. So for $c \in Cut \cup \{\top\}$ we set $area_{\mathfrak{E}}(c) := in(c) \backslash \bigcup_{d<c} in(d)$.
$\mathbb{R}^2$ is the disjoint union of all sets $area_{\mathfrak{E}}(c)$. So we can define a mapping $cut_{\mathfrak{E}} : \mathbb{R}^2 \to Cut \cup \{\top\}$ with $\vec{x} \in area_{\mathfrak{E}}(cut_{\mathfrak{E}}(\vec{x}))$ for each $\vec{x} \in \mathbb{R}^2$.
Finally we set $Hook := \{(\vec{x_p}, i) \mid p \in P \wedge 1 \leq i \leq arity(p)\}$, $PrSpot := \{\vec{x_p} \in \mathbb{R}^2 \mid p \in P\}$ and $IdSpot := \{\vec{x} \in \mathbb{R}^2 \mid \exists l \in L. \exists x \in \{0,1\}. \nu_l(x) = \vec{x}\} \backslash PrSpot$.

**Definition 3.** *Let $\mathfrak{E} := (L, (\nu_l)_{l \in L}, \top, Cut, (\nu_c)_{c \in Cut}, P, (\nu_p)_{p \in P})$ be a formal instance of an existential graph. Then let $\Xi(\mathfrak{E}) := (V, E, \nu, \top, Cut, area, \kappa, \rho)$ be the following concept graph with cuts:*
- *$V := IdSpot \,\dot\cup\, Hook$ and $E := P \,\dot\cup\, L$ ,*

– $\nu$ *is defined as follows:*
  - *For* $p \in P$ *we set* $\nu(p) := ((\vec{x_p}, 1), \ldots, (\vec{x_p}, arity(p))$
  - *For* $l \in L$ *we set* $\nu(l) := (\nu_{l,0}, \nu_{l,0})$ *with*

$$\nu_{l,x} := \begin{cases} \vec{x} & \text{if } \nu_l(x) \in IdSpot \\ (\vec{x}, i) & \text{if there is a } p \in P \text{ with } l_{p,i} = l \text{ and } x_{p,i} = x \end{cases} ,$$

– *For* $c \in Cut \cup \{\top\}$ *we set*

$$area(c) := \{d \in Cut \,|\, \nu_d[S^1] \subseteq area_E G(c)\} \,\dot\cup\, \{\vec{x} \in IDSpot \,|\, cut_{\mathfrak{E}}(\vec{x}) = c\} \,\dot\cup\, \{(\vec{x}, i) \in Hook \,|\, cut_{\mathfrak{E}}(\vec{x}) = c\} \,\dot\cup\, \{l \in L \,|\, cut_{\mathfrak{E}}(\nu_l(\tfrac{1}{2})) = c\} ,$$

– $\kappa(v) := \top$ *for* $v \in V$, $\kappa(p) := R_p$ *for* $p \in P$, $\kappa(l) := id$ *for* $l \in L$, *and*
– $\rho(v) := \top$ *for all* $v \in V$.

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